

# Pricing American options using martingale bases

Jérôme Lelong\*

*Laboratoire Jean Kuntzmann, Université Grenoble Alpes, FRANCE.*

jerome.lelong@imag.fr

April 13, 2016

## Abstract

In this work, we propose an algorithm to price American options by directly solving the dual minimization problem introduced by Rogers [2002]. Our approach relies on approximating the set of uniformly square integrable martingales by a finite dimensional Wiener chaos expansion. Then, we use a sample average approximation technique to efficiently solve the optimization problem. Unlike all the regression based methods, our method can transparently deal with path dependent options without extra computations and a parallel implementation writes easily with very little communication and no centralized work. We test our approach on several multi-dimensional options with up to 40 assets and show the impressive scalability of the parallel implementation.

**Key words:** American option, duality, Snell envelope, stochastic optimization, sample average approximation, high performance computing, Wiener chaos expansion.

**AMS subject classification:** 62L20, 62L15, 91G60, 65Y05, 60H07

## 1 Introduction

The pricing of American options quickly becomes challenging as the dimension increases and the payoff gets complex. Many people have contributed to this problem usually by considering its dynamic programming principle formulation Tilley [1993], Carriere [1996], Tsitsiklis and Roy [2001], Longstaff and Schwartz [2001], Broadie and Glasserman [2004] and Bally and Pages [2003]. Among this so extensive literature, the practitioners seem to prefer the iterative optimal policy approach proposed by Longstaff and Schwartz [2001], which proves to be quite efficient in many situations. However, *true* path-dependent options cannot be handled by this approach. Solving the dynamic programming principle requires the computation of a conditional expectation, which is eventually dealt with regression

---

\*This project was supported by the Finance for Energy Market Research Centre, [www.fime-lab.org](http://www.fime-lab.org). The High Performance Computations presented in this paper were performed using the Froggy platform of the CIMENT infrastructure (<https://ciment.ujf-grenoble.fr>), which is supported by the Rhône-Alpes region (GRANT CPER07\_13 CIRA) and the Equip@Meso project (reference ANR-10-EQPX-29-01) of the programme Investissements d'Avenir supervised by the Agence Nationale pour la Recherche.

techniques. These techniques are known to suffer from the curse of dimensionality: global regression methods lead to high dimensional linear algebra problems, whereas local methods see the number of domains blow up with the dimension. Despite the numerous parallel implementation of this techniques (see for instance Dung Doan et al. [2010], Abbas-Turki et al. [2014]), we cannot expect to obtain a fully scalable algorithm. In this work, we follow the dual approach initiated by Rogers [2002], and Davis and Karatzas [1994], which can naturally handle path dependent options. To make it implementable, we need a smart and finite dimensional approximation of the set of uniformly integrable martingales. We chose the set of truncated Wiener chaos expansions, which have some magic features in our problem: it regularizes the optimization problem and computing its conditional expectation exactly is straightforward. Then, the pricing problem boils down finite dimensional, convex and differentiable optimization problem. The optimization problem is solved using a *Sample Average Approximation* (see Rubinstein and Shapiro [1993]), which can be easily and efficiently implemented using parallel computing.

We fix some finite time horizon  $T > 0$  and a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ , where  $(\mathcal{F}_t)_{0 \leq t \leq T}$  is supposed to be the natural augmented filtration of a  $d$ -dimensional Brownian motion  $B$ . On this space, we consider an adapted process  $(S_t)_{0 \leq t \leq T}$  with values in  $\mathbb{R}^{d'}$  modeling a  $d'$ -dimensional underlying asset. The number of assets  $d'$  can be smaller than the dimension  $d$  of the Brownian motion to encompass the case of stochastic volatility models or stochastic interest rate. We assume that the short interest rate is modelled by an adapted process  $(r_t)_{0 \leq t \leq T}$  with values in  $\mathbb{R}_+$  and that  $\mathbb{P}$  is an associated risk neutral measure. We consider an adapted payoff process  $\tilde{Z}$  and introduce its discounted value process  $\left( Z_t = e^{-\int_0^t r_s ds} \tilde{Z}_t \right)_{0 \leq t \leq T}$ . We assume that the paths of  $Z$  are right continuous and that  $\sup_{t \in [0, T]} |Z_t| \in L^2$ . The process  $\tilde{Z}$  can obviously take the simple form  $(\phi(S_t))_{t \leq T}$  but it can also depend on the whole path of  $S$  up to the current time. So, our framework transparently deals with path-dependent option, which are far more difficult to handle using regression techniques.

We consider the American option paying  $\tilde{Z}_t$  to its holder if exercised at time  $t$ . Standard arbitrage pricing theory defines the discounted time- $t$  value of the American option to be

$$U_t = \text{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}[Z_\tau | \mathcal{F}_{t_k}] \quad (1)$$

where  $\mathcal{T}_t$  denotes the set of  $\mathcal{F}$ -stopping times with values in  $[t, T]$ . The integrability properties of  $Z$  ensure that  $U$  is a supermartingale of class (D) and hence has a Doob–Meyer decomposition

$$U_t = U_0 + M_t^* - A_t^* \quad (2)$$

where  $M^*$  is a martingale vanishing at zero and  $A^*$  is a predictable integrable increasing process also vanishing at zero. With our assumptions on  $Z$ ,  $M^*$  is square integrable. Rogers [2002] found an alternative representation of the price at time-0 of the American option as the minimum value of the following optimization problem

$$U_0 = \inf_{M \in H_0^2} \mathbb{E} \left[ \sup_{t \leq T} (Z_t - M_t) \right] = \mathbb{E} \left[ \sup_{t \leq T} (Z_t - M_t^*) \right] \quad (3)$$

where  $H_0^2$  denotes the set of square integrable martingales vanishing at zero. A martingale reaching the infimum is called an *optimal* martingale. As the dual price problem writes

as a convex minimisation problem, the set of all optimal martingales is a convex subset of  $H_0^2$ . Among the martingales reaching the infimum in (3), some of them actually satisfy the pathwise equality  $\sup_{t \leq T} Z_t - M_t = U_0$ . These martingales are called *surely optimal*. Any surely optimal martingale reaches the lower bound in (3) but not all optimal martingales are surely optimal. We refer to Schoenmakers et al. [2013] for a detailed characterisation of optimal martingales. Anyway, Jamshidian [2007] proved the uniqueness of surely optimal martingales within the continuing region, ie. for any surely optimal martingale  $M$  and any optimal strategy  $\tau$ ,  $(M_{t \wedge \tau})_t = (M_{t \wedge \tau}^*)_t$  a.s.

The most famous method using the dual representation (3) is probably the primal–dual approach of Andersen and Broadie [2004], which heavily relies on the knowledge of an optimal exercising policy. The a priori knowledge may take the form of nested Monte Carlo simulations as in Schoenmakers [2005], and Kolodko and Schoenmakers [2004]. To circumvent this difficulty, Rogers [2010] explained how to construct a *good* martingale. In a Wiener framework, Belomestny et al. [2009] investigated this approach by relying on the martingale representation theorem to build *good* martingales. When trying to practically use the dual formulation (3), the first difficulty is to find a rich enough but finite dimensional approximation of  $H_0^2$  and then we face a finite although potentially high–dimensional minimization problem (see Belomestny [2013] for one way of handling this approach).

The minimization problem (3) can be equivalently formulated as

$$U_0 = \inf_{X \in L_0^2(\Omega, \mathcal{F}_T, \mathbb{P})} \mathbb{E} \left[ \sup_{0 \leq t \leq T} (Z_t - \mathbb{E}[X | \mathcal{F}_t]) \right] \quad (4)$$

where  $L_0^2(\Omega, \mathcal{F}_T, \mathbb{P})$  is the set of square integrable  $\mathcal{F}_T$ – random variables with zero mean. In this work, we suggest to use the truncated Wiener chaos expansion as a finite dimensional approximation of  $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ . Since Wiener chaos are orthogonal for the  $L^2$  inner product, the computations of the conditional expectations  $\mathbb{E}[X | \mathcal{F}_t]$  become straightforward and boil down to dropping some terms in the chaos expansion, which makes our approach very convenient. Based on this approximation, we propose a scalable algorithm and study its convergence.

The paper starts with the presentation of the Wiener chaos expansion and some of its useful properties in Section 2. Then, we can develop the core of our work in Section 3 in which we explain how the price of the American option can be approximated by the solution of a finite dimensional optimization problem. First, we analyze the properties of the optimization problem in order to prove the convergence of its solution to the American option price. Second, we study its sample average approximation, which makes the problem tractable, and prove its convergence. Based on all these theoretical results, we present our algorithm in Section 4 and discuss its parallel implementation on distributed memory architectures. Finally, some numerical examples are presented in Section 5.

## Notation

- For  $n \geq 1$ ,  $0 = t_0 < t_1 < \dots < t_n = T$  is a time grid of  $[0, T]$  satisfying  $\lim_{n \rightarrow \infty} \sup_{0 \leq k \leq n-1} |t_{k+1} - t_k| = 0$ .
- For  $n \geq 1$ , the discrete time filtration  $\mathcal{G}$  is defined by  $\mathcal{G}_k = \sigma(B_{t_{i+1}} - B_{t_i}, i = 0, \dots, k-1)$  for all  $1 \leq k \leq n$ , while  $\mathcal{G}_0$  is the trivial sigma algebra. Obviously,  $\mathcal{G}_k \subset \mathcal{F}_{t_k}$  for all  $0 \leq k \leq n$ .

- For  $1 \leq q \leq d$ ,  $\mathbb{I}(r) \in \{0, 1\}^n$  denotes the vector  $(\underbrace{0, \dots, 0}_{r-1}, 1, \underbrace{0, \dots, 0}_{n-r})$ .
- For  $1 \leq q \leq d$ , and  $1 \leq r \leq n$ ,  $\mathbb{I}(r, q) \in \mathbb{N}^{n \times d}$  with all components equal to 0 except the component with index  $(r, q)$  which is equal to 1.

## 2 Wiener chaos expansion

For the sake of clearness, we first present the Wiener chaos expansion in the case  $d = 1$  (ie.  $B$  is a real valued Brownian motion).

### 2.1 General framework in dimension one

**Iterated stochastic integral approach.** For a sequence of deterministic functions  $(f_n)_{n \geq 1}$  such that  $h_n : [0, T]^n \rightarrow \mathbb{R}$  and  $\int_{[0, T]^n} |f_n(t)|^2 dt < \infty$ , we define the iterated stochastic integral by

$$\begin{cases} V_1(f_1) &= \int_0^T f_1(t_1) dB_{t_1} \\ V_n(f_n) &= \int_0^T \int_0^{t_n} \dots \int_0^{t_2} f_n(t_n, \dots, t_1) dB_{t_1} \dots dB_{t_n}, \quad \forall n \geq 2 \end{cases} \quad (5)$$

The set  $\{V_n(f) : f \in L^2([0, T]^n)\}$  is a subspace of  $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ , whose closure is often referred to as the Wiener chaos of order  $n$ .

We know from Nualart [1998] that any square integrable, real valued and  $\mathcal{F}_T$ -measurable random variable  $F$  can be expanded as a series of iterated stochastic integrals

$$F = \mathbb{E}[F] + \sum_{n \geq 1} V_n(f_n) \quad (6)$$

where the deterministic functions  $(f_i)_{i \geq 1}$  are symmetric. We define the chaos expansion of order  $p \geq 1$  as

$$C_p(F) = \mathbb{E}[F] + \sum_{n=1}^p V_n(f_n) \quad (7)$$

which corresponds to the truncation of the sum in Equality (6) to  $p$  terms. For instance, for  $p = 2$ ,  $C_2(F)$  only involves a Wiener integral and a double stochastic integral apart from the constant term  $\mathbb{E}[F]$ .

**Hermite polynomials approach.** The iterated stochastic integral approach to the Wiener chaos expansion is not applicable in practice and cannot be generalized to multi-dimensional Brownian motions. Hopefully, this expansion can be formulated in terms of Hermite polynomials.

Let  $H_i$  be the  $i$ -th Hermite polynomial defined by

$$H_0(x) = 1; \quad H_i(x) = (-1)^i e^{x^2/2} \frac{d^i}{dx^i} (e^{-x^2/2}), \text{ for } i \geq 1. \quad (8)$$

They satisfy for all integer  $i$ ,  $H'_i = H_{i-1}$  with the convention  $H_{-1} = 0$ . We recall that if  $(X, Y)$  is a random normal vector with  $\mathbb{E}[X] = \mathbb{E}[Y] = 0$  and  $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$

$$\mathbb{E}[H_i(X)H_j(Y)] = i! (\mathbb{E}[XY])^i \mathbf{1}_{i=j}. \quad (9)$$

For all  $i \geq 0$ , we define the spaces

$$\mathcal{H}_i = \left\{ H_i \left( \int_0^T f_t dB_t \right) : f \in L^2([0, T]) \right\} \quad (10)$$

whose  $L^2$  closure corresponds to the Wiener chaos of order  $i$ .

We consider the indicator functions of the grid defined by  $t_0 < t_1 < \dots < t_n$

$$f_i(t) = \mathbf{1}_{]t_{i-1}, t_i]}(t) / \sqrt{t_i - t_{i-1}}, \quad i = 1, \dots, n, \quad (11)$$

With this choice for the  $(f_i)_i$ ,

$$\int_0^T f_i(t) dB_t = \frac{B_{t_i} - B_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}} = G_i.$$

Note that the random variables  $G_i$  are i.i.d. following the standard normal distribution. We complete these  $n$  functions  $f_1, \dots, f_n$  into an orthonormal basis of  $L^2([0, T])$  and introduce the truncated chaos expansion of order  $p$  on the basis of  $L^2$

$$C_{p,n}(F) = \sum_{\alpha \in A_{p,n}} \lambda_\alpha \prod_{i \geq 1} H_{\alpha_i}(G_i) \quad (12)$$

where  $A_{p,n} = \{\alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq p\}$  with  $\|\alpha\|_1 = \sum_{i \geq 0} \alpha_i$ . Using Equation (9), we deduce that the coefficients of the above decomposition are uniquely determined by

$$\lambda_\alpha = \frac{\mathbb{E} \left[ F \prod_{i \geq 1} H_{\alpha_i}(G_i) \right]}{\left( \prod_{i \geq 1} \alpha_i! \right)}. \quad (13)$$

This formula can be rewritten more clearly by introducing the generalized Hermite polynomials defined for any multi-index  $\alpha = (\alpha_i)_{i \geq 1} \in \mathbb{N}^{\mathbb{N}}$

$$\hat{H}_\alpha(x) = \prod_{i \geq 1} H_{\alpha_i}(x_i), \quad \text{for } x \in \mathbb{R}^{\mathbb{N}}. \quad (14)$$

With this notation, Equation (12) becomes

$$C_{p,n}(F) = \sum_{\alpha \in A_{p,n}} \lambda_\alpha \hat{H}_\alpha(G_1, \dots, G_n).$$

**Proposition 2.1.** *Let  $F$  be a real valued random variable in  $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$  and let  $k \in \{1, \dots, n\}$  and  $p \geq 0$*

$$\mathbb{E}[C_{p,n}(F) | \mathcal{F}_{t_k}] = \sum_{\alpha \in A_{p,n}^k} \lambda_\alpha \hat{H}_\alpha(G_1, \dots, G_n)$$

with  $A_{p,n}^k = \{\alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq p, \alpha_\ell = 0 \forall \ell > k\}$ .

*Proof.* Taking the conditional expectation in Eq. (12) leads to

$$\mathbb{E}[C_{p,n}(F) | \mathcal{F}_{t_k}] = \sum_{\alpha \in A_{p,n}} \lambda_\alpha \prod_{i=1}^k H_{\alpha_i}(G_i) \mathbb{E} \left[ \prod_{i=k+1}^n H_{\alpha_i}(G_i) \middle| \mathcal{F}_{t_k} \right]. \quad (15)$$

Since the Brownian increments after time  $t_k$  are independent of  $\mathcal{F}_{t_k}$  and are independent of one another,  $\mathbb{E} \left[ \prod_{i=k+1}^n H_{\alpha_i}(G_i) | \mathcal{F}_{t_k} \right] = \prod_{i=k+1}^n \mathbb{E}[H_{\alpha_i}(G_i)]$ , which is zero as soon as  $\sum_{i=k+1}^n \alpha_i > 0$ . Hence, the sum in Equation (15) is reduced to the sum over the set of multi-indices  $\alpha \in A_{p,n}$  such that  $\alpha_i = 0$  for all  $i > k$ , which is exactly the definition of the set  $A_{p,n}^k$ . ■

**Remark 2.2.** Since the sum appearing in  $\mathbb{E}[C_{p,n}(F)|\mathcal{F}_{t_k}]$  is reduced to a sum over the set of multi-indices  $\alpha \in A_{p,n}^k$ , it actually only depends on the first  $k$  increments  $(G_1, \dots, G_k)$ . One can easily check that  $\mathbb{E}[C_{p,n}(F)|\mathcal{F}_{t_k}]$  is actually given by the chaos expansion of  $F$  on the first  $k$  Brownian increments. Hence, computing a conditional expectation simply boils down to dropping term. While it may look like a naive way to proceed, it is indeed correct in our setting.

**Proposition 2.3.** *Let  $F$  be a real valued random variable in  $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$  and let  $k \in \{1, \dots, n\}$  and  $p \geq 1$ . For all  $t \in ]t_{r-1}, t_r]$  with  $1 \leq r \leq k$ ,*

$$D_t \mathbb{E}[C_{p,n}(F)|\mathcal{F}_{t_k}] = \frac{1}{\sqrt{h}} \sum_{\alpha \in A_{p,n}^k, \alpha_r \geq 1} \lambda_\alpha \hat{H}_{\alpha - \mathbb{I}(r)}(G_1, \dots, G_n)$$

where  $\alpha - \mathbb{I}(r) = (\alpha_1, \dots, \alpha_{r-1}, \alpha_r - 1, \alpha_{r+1}, \dots, \alpha_n)$ .

*Proof.* From Proposition 2.1, we know that for all  $1 \leq k \leq n$

$$\mathbb{E}[C_{p,n}(F)|\mathcal{F}_{t_k}] = \sum_{\alpha \in A_{p,n}^k} \lambda_\alpha \hat{H}_\alpha(G_1, \dots, G_n)$$

Let  $r \leq k$  and  $t \in ]t_{r-1}, t_r]$ . The chain rule for the Malliavin derivative yields

$$\begin{aligned} D_t \mathbb{E}[C_{p,n}(F)|\mathcal{F}_{t_k}] &= \sum_{\alpha \in A_{p,n}^k} \lambda_\alpha D_t \left( \prod_{i=1}^k H_{\alpha_i}(G_i) \right) \\ D_t \left( \prod_{i=1}^k H_{\alpha_i}(G_i) \right) &= \prod_{i=1, i \neq r}^k H_{\alpha_i}(G_i) H'_{\alpha_r}(G_r) \\ &= \mathbf{1}_{\alpha_r \geq 1} \prod_{i=1, i \neq r}^k H_{\alpha_i}(G_i) H_{\alpha_r - 1}(G_r) \\ &= \mathbf{1}_{\alpha_r \geq 1} \hat{H}_{\alpha - \mathbb{I}(r)}(G_1, \dots, G_n). \end{aligned} \quad \blacksquare$$

## 2.2 Multi-dimensional chaos expansion

In the previous section, we explained how a random variable measurable for a sigma field generated by a one-dimensional Brownian motion could be approximated by a finite sum of Hermite polynomials of Brownian increments.

In this section, we are back to our original multi-dimensional setting, as explained in Section 1. The process  $B$  is a Brownian motion with values in  $\mathbb{R}^d$ . The key idea to extend the Hermite polynomial expansion to a higher dimensional setting is to consider a tensor product of Hermite polynomials evaluated on a tensor basis of  $L^2([0, T], \mathbb{R}^d)$ .

Consider the functions  $(h_i)_i$  with values in  $\mathbb{R}^d$  defined by

$$h_i^j(t) = \frac{\mathbf{1}_{]t_{i-1}, t_i]}(t)}{\sqrt{h}} \mathbf{e}_j, \quad i = 1, \dots, n, \quad j = 1, \dots, d$$

where  $(\mathbf{e}_1, \dots, \mathbf{e}_d)$  denotes the canonical basis of  $\mathbb{R}^d$ . The  $p$ -th order Wiener chaos  $\mathcal{C}_{p,n}$  is defined as the closure of

$$\left\{ \prod_{j=1}^d \hat{H}_{\alpha^j}(G_1^j, \dots, G_n^j) : \alpha \in (\mathbb{N}^n)^d, \|\alpha\|_1 \leq p \right\}$$

where  $\|\alpha\|_1 = \sum_{i=1}^n \sum_{j=1}^d \alpha_i^j$  and  $G_i^j = \frac{B_{t_i}^j - B_{t_{i-1}}^j}{\sqrt{h}}$ . Using the independence of the Brownian increments and the orthogonality of the Hermite polynomials, the chaos expansion of a square integrable random variable  $F$  is given by

$$C_{p,n}(F) = \sum_{\alpha \in A_{p,n}^{\otimes d}} \lambda_\alpha \hat{H}_\alpha^{\otimes d}(G_1, \dots, G_n)$$

where

$$\begin{aligned} \hat{H}_\alpha^{\otimes d}(G_1, \dots, G_n) &= \prod_{j=1}^d \hat{H}_{\alpha^j}(G_1^j, \dots, G_n^j) \quad \forall \alpha \in (\mathbb{N}^n)^d \\ A_{p,n}^{\otimes d} &= \left\{ \alpha \in (\mathbb{N}^n)^d : \|\alpha\|_1 \leq p \right\}. \end{aligned} \quad (16)$$

With an obvious abuse of notation, we write, for  $\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}}$ ,

$$C_{p,n}(\lambda) = \sum_{\alpha \in A_{p,n}^{\otimes d}} \lambda_\alpha \hat{H}_\alpha^{\otimes d}(G_1, \dots, G_n).$$

We also introduce the set of multi-indices truncated after time  $t_k$

$$A_{p,n}^{\otimes d,k} = \left\{ \alpha \in A_{p,n}^{\otimes d} : \forall j \in \{1, \dots, d\}, \forall \ell > k, \alpha_\ell^j = 0 \right\}. \quad (17)$$

We introduce the set  $\mathcal{C}_{p,n}$  defined by

$$\mathcal{C}_{p,n} = \left\{ F \in L^2(\Omega, \mathcal{F}_T, P) : F = C_{p,n}(F) \text{ a.s.} \right\}.$$

We can easily deduce the multidimensional counterpart of Proposition 2.1

**Proposition 2.4.** *Let  $F$  be a real valued random variable in  $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$  and let  $k \in \{1, \dots, n\}$  and  $p \geq 0$*

$$\mathbb{E}[C_{p,n}(F) | \mathcal{F}_{t_k}] = \sum_{\alpha \in A_{p,n}^{\otimes d,k}} \lambda_\alpha \hat{H}_\alpha^{\otimes d}(G_1, \dots, G_n).$$

**Remark 2.5.** The discrete time sequence  $(\mathbb{E}[C_{p,n}(F) | \mathcal{F}_{t_k}])_{0 \leq k \leq n}$  is of course adapted to the filtration  $(\mathcal{F}_{t_k})_k$  but also to the smaller filtration  $(\mathcal{G}_k)_k$ . This property plays a crucial when approximating a random variable  $F \in L^2(\Omega, \mathcal{G}_n, \mathbb{P})$  as we know from [Nualart, 1998, Theorem 1.1.1] that in such a case  $\lim_{p \rightarrow \infty} C_{p,n}(F) = F$  in the  $L^2$ -sense. This result holds for the fixed value  $n$ . If  $F$  were only  $\mathcal{F}_T$ -measurable and not  $\mathcal{G}_n$ -measurable, we would need to impose that  $F \in \mathbb{D}^{1,2}$  to obtain  $\lim_{p \rightarrow \infty, n \rightarrow \infty} C_{p,n}(F) = F$ . In this latter case, it is required to let  $n$  go to infinity to recover  $F$ .

**Proposition 2.6.** *Let  $F$  be a real valued random variable in  $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$  and let  $k \in \{1, \dots, n\}$  and  $p \geq 1$ . For  $t > t_k$ ,  $D_t \mathbb{E}[C_{p,n}(F) | \mathcal{F}_{t_k}] = 0$ .*

*For all  $t \in [t_{r-1}, t_r]$  with  $1 \leq r \leq k$ , and  $q = 1, \dots, d$ ,*

$$D_t^q \mathbb{E}[C_{p,n}(F) | \mathcal{F}_{t_k}] = \frac{1}{\sqrt{h}} \sum_{\alpha \in A_{p,n}^{\otimes d,k}, \alpha_r^q \geq 1} \lambda_\alpha \hat{H}_{\alpha - \mathbb{I}(r,q)}^{\otimes d}(G_1, \dots, G_n)$$

where  $(\alpha - \mathbb{I}(r,q))_i^j = \alpha_i^j - \mathbf{1}_{j=q, i=r}$ .

In this multi-dimensional setting, the Malliavin derivative operator is actually a gradient operator  $D_t = (D_t^1, \dots, D_t^d)$ .

*Proof.* From Proposition 2.4, we know that for all  $1 \leq k \leq n$

$$\mathbb{E}[C_{p,n}(F)|\mathcal{F}_{t_k}] = \sum_{\alpha \in A_{p,n}^{\otimes d,k}} \lambda_\alpha \prod_{j=1}^d \hat{H}_{\alpha^j}(G_1^j, \dots, G_n^j).$$

Let  $r \leq k$  and  $t \in ]t_{r-1}, t_r]$ . Let  $1 \leq q \leq d$ . The chain rule for the Malliavin derivative yields

$$\begin{aligned} D_t^q \mathbb{E}[C_{p,n}(F)|\mathcal{F}_{t_k}] &= \sum_{\alpha \in A_{p,n}^{\otimes d,k}} \lambda_\alpha D_t^q \left( \prod_{j=1}^d \hat{H}_{\alpha^j}(G_1^j, \dots, G_n^j) \right) \\ &= \sum_{\alpha \in A_{p,n}^{\otimes d,k}} \lambda_\alpha \left( \prod_{j=1, j \neq q}^d \hat{H}_{\alpha^j}(G_1^j, \dots, G_n^j) \right) D_t^q \left( \hat{H}_{\alpha^q}(G_1^q, \dots, G_n^q) \right) \\ &= \frac{1}{\sqrt{h}} \sum_{\alpha \in A_{p,n}^{\otimes d,k}} \lambda_\alpha \left( \prod_{j=1, j \neq q}^d \hat{H}_{\alpha^j}(G_1^j, \dots, G_n^j) \right) \hat{H}_{\alpha^q - \mathbb{I}(r)}(G_1^q, \dots, G_n^q) \\ &= \frac{1}{\sqrt{h}} \sum_{\alpha \in A_{p,n}^{\otimes d,k}, \alpha_r^q \geq 1} \lambda_\alpha \hat{H}_{\alpha - \mathbb{I}(r,q)}(G_1, \dots, G_n). \end{aligned} \quad \blacksquare$$

**Remark 2.7.** The conditional expectation preserves the nature of a chaos expansion. Similarly, the Malliavin derivative of a chaos expansion still writes as a chaos expansion and hence is a Hermite polynomial of Brownian increments. The roots of a non zero polynomial being a zero measure set and since the Brownian increments have a joined density, the Malliavin derivative of a chaos expansion is almost surely non zero as soon as one of the coefficients  $\lambda_\alpha$  is non zero for  $\alpha \in A_{p,n}^{\otimes d,k}$  such that  $\alpha_r^j \geq 1$  for some  $j \in \{1, \dots, d\}$ .

For  $i, k \in \{1, \dots, n\}$ , with  $i < k$ , we introduce the set  $A_{p,n}^{\otimes d,i:k}$  defined as  $A_{p,n}^{\otimes d,k} \setminus A_{p,n}^{\otimes d,i}$ .

$$A_{p,n}^{\otimes d,i:k} = \left\{ \alpha \in (\mathbb{N}^n)^d : \|\alpha\|_1 \leq p, \text{ and } \forall 1 \leq j \leq d, \forall \ell \notin \{i+1, \dots, k\}, \alpha_\ell^j = 0 \right\}. \quad (18)$$

### 3 Pricing American options using Wiener chaos expansion and sample average approximation

In this section, we aim at approximating the dual price (4) by a tractable optimization problem. This involves two kinds of approximations: first, approximate the space  $L_0^2(\Omega, \mathcal{F}_T, \mathbb{P})$  by a finite dimensional vector space; second, replace the expectation by a sample average approximation.

The dual price writes

$$\inf_{X \in L_0^2(\Omega, \mathcal{F}_T, \mathbb{P})} \mathbb{E} \left[ \sup_{0 \leq t \leq T} (Z_t - \mathbb{E}[X|\mathcal{F}_t]) \right].$$

In this optimization problem, we replace  $X$  by its chaos expansion  $C_{p,n}(X)$ , which has no constant term as  $\mathbb{E}[X] = 0$  and we approximate the supremum by a discrete time maximum.



Then, we face a finite dimensional minimization problem to determine the optimal solution with the subset  $\mathcal{C}_{p,n}$

$$\inf_{\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}}, \lambda_0=0} \mathbb{E} \left[ \max_{0 \leq k \leq n} (Z_{t_k} - \mathbb{E}[C_{p,n}(\lambda) | \mathcal{F}_{t_k}]) \right]. \quad (19)$$

In Section 3.1, we prove that this optimization problem is convex and has a solution (see Proposition 3.1) and converges to the price of the American option (see Proposition 3.2). Moreover, as the cost function is differentiable, any minimizer is a zero of the gradient (see Proposition 3.5), which makes it easier to derive an algorithm.

To come up with a fully implementable algorithm, Section 3.2 presents the sample average approximation of (19), which consists in replacing the expectation by a Monte Carlo summation. We prove in Proposition 3.6 that the solution of the sample average approximation converges to the solution of (19) when the number of samples goes to infinity.

### 3.1 A stochastic optimization approach

We fix  $p \geq 1$  and define the random functions  $v_{p,n}(\cdot, \cdot; Z, G) : \mathbb{R}^{A_{p,n}^{\otimes d}} \times \{0, \dots, n\}$  by

$$v_{p,n}(\lambda, k; Z, G) = Z_{t_k} - \sum_{\alpha \in A_{p,n}^{\otimes d}} \lambda_{\alpha} \mathbb{E} \left[ \hat{H}_{\alpha}^{\otimes d}(G_1, \dots, G_n) \middle| \mathcal{F}_{t_k} \right],$$

With the help of Proposition 2.1, the random functions  $v_{p,n}$  can be rewritten

$$v_{p,n}(\lambda, k, Z, G) = Z_{t_k} - \sum_{\alpha \in A_{p,n}^{\otimes d,k}} \lambda_{\alpha} \hat{H}_{\alpha}^{\otimes d}(G_1, \dots, G_n). \quad (20)$$

We consider the cost function  $V_{p,n} : \mathbb{R}^{A_{p,n}^{\otimes d}} \rightarrow \mathbb{R}$  defined by

$$V_{p,n}(\lambda) = \mathbb{E} \left[ \max_{0 \leq k \leq n} v_{p,n}(\lambda, k; Z, G) \right] \quad (21)$$

and we approximate the solution of (4) by

$$\inf_{\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}}, \lambda_0=0} V_{p,n}(\lambda). \quad (22)$$

We introduce the set of random indices for which the pathwise maximum is attained

$$\mathcal{I}(\lambda, Z, G) = \left\{ 0 \leq k \leq n : v_{p,n}(\lambda, k; Z, G) = \max_{\ell \leq n} v_{p,n}(\lambda, \ell; Z, G) \right\}. \quad (23)$$

**Proposition 3.1.** *The minimization problem (22) has at least one solution.*

*Proof.* As the supremum of linear functions is convex, the random function  $\lambda \mapsto \max_{k \leq n} v_{p,n}(\lambda, t_k, Z, G)$  is almost surely convex. The convexity of  $V_{p,n}$  ensues from the linearity of the expectation.

Let us prove that  $V_{p,n}(\lambda) \rightarrow \infty$  when  $|\lambda| \rightarrow \infty$ . Note that  $V_{p,n}(\lambda) \geq \mathbb{E}[(C_{p,n}(\lambda))_-] \geq \frac{1}{2} \mathbb{E}[|C_{p,n}(\lambda)|]$ , where we have used that  $|x| = 2x_- + x$  and  $\mathbb{E}[C_{p,n}(\lambda)] = 0$ .

$$\mathbb{E}[|C_{p,n}(\lambda)|] = |\lambda| \mathbb{E}[|C_{p,n}(\lambda/|\lambda|)|] \geq |\lambda| \inf_{\mu \in \mathbb{R}^{A_{p,n}^{\otimes d}}, |\mu|=1} \mathbb{E}[|C_{p,n}(\mu)|].$$

By a standard continuity argument, the infimum is attained. Moreover, it is strictly positive as otherwise there would exist  $\mu \in \mathbb{R}^{A_{p,n}^{\otimes d}}$  with  $|\mu| = 1$  s.t.  $\mathbb{E}[|C_{p,n}(\mu)|] = 0$ . Using the orthogonality of the family  $(H_\alpha^{\otimes d})_{\alpha \in A_{p,n}^{\otimes d}}$ , we would immediately deduce that  $\mu = 0$ . Hence, we show that  $V_{p,n}(\lambda) \rightarrow \infty$  when  $|\lambda| \rightarrow \infty$ . The growth at infinity of  $V_{p,n}$  combined with its convexity yields the existence of a solution to the minimization problem (22). ■

Proposition 3.1 ensures the existence of  $\lambda_{p,n}^\sharp$  solving (22), ie.

$$V_{p,n}(\lambda_{p,n}^\sharp) = \inf_{\lambda \text{ s.t. } \lambda_0=0} V_{p,n}(\lambda). \quad (24)$$

Moreover,  $\nabla V_{p,n}(\lambda_{p,n}^\sharp) = 0$ . This characterization of an optimal solution will be of prime importance to practically devise an algorithm.

**Proposition 3.2.** *The solution of the minimization problem (22),  $V_{p,n}(\lambda_{p,n}^\sharp)$ , converges to  $U_0$  when both  $p$  and  $n$  go to infinity.*

*Proof.* We introduce the truncated chaos expansion of  $M_T^*$  and denote its coefficients by  $\lambda_{p,n}^*$ , ie.  $C_{p,n}(M_T^*) = C_{p,n}(\lambda_{p,n}^*)$ . Clearly,  $U_0 \leq V_{p,n}(\lambda_{p,n}^\sharp) \leq V_{p,n}(\lambda_{p,n}^*)$ . Then, we obtain the following result

$$\begin{aligned} 0 &\leq V_{p,n}(\lambda_{p,n}^\sharp) - U_0 \leq V_{p,n}(\lambda_{p,n}^*) - U_0 \\ &= \mathbb{E} \left[ \max_k (Z_{t_k} - \mathbb{E}[C_{p,n}(\lambda_{p,n}^*) | \mathcal{F}_{t_k}]) - \max_k (Z_{t_k} - M_{t_k}^*) \right] \\ &\leq \mathbb{E} \left[ \max_k \left| M_{t_k}^* - \mathbb{E}[C_{p,n}(\lambda_{p,n}^*) | \mathcal{F}_{t_k}] \right| \right] \\ &\leq \mathbb{E} \left[ \max_k \mathbb{E} \left[ \left| M_T^* - C_{p,n}(\lambda_{p,n}^*) \right| | \mathcal{F}_{t_k} \right] \right] \\ &\leq \sqrt{\mathbb{E} \left[ \max_k \mathbb{E} \left[ \left| M_T^* - C_{p,n}(\lambda_{p,n}^*) \right|^2 | \mathcal{F}_{t_k} \right] \right]} \\ &\leq 2 \|M_T^* - C_{p,n}(M_T^*)\|_2 \end{aligned} \quad (25)$$

where the last upper-bound ensues from Doob's inequality. Note that this bound does not depend on  $\lambda_{p,n}^\sharp$ . The convergence result when  $p, n$  go to infinity ensues from [Briand and Labart, 2014, Lemma 2, Lemma 19]. ■

**Remark 3.3.** Note that if in the series of inequalities (25), we had dropped the second one, we would have come up in the end with  $0 \leq V_{p,n}(\lambda_{p,n}^\sharp) - U_0 \leq 2 \|M_T^* - C_{p,n}(\lambda_{p,n}^\sharp)\|_2$ , which, by definition of the chaos expansion, is larger than  $2 \|M_T^* - C_{p,n}(M_T^*)\|_2$ .

**Corollary 3.4.** *Consider the Bermudean option with exercising dates  $t_0, \dots, t_n$  and with discounted payoff  $(Z_{t_k})_k$  assumed to be  $\mathcal{G}$ -adapted. Then,  $V_{p,n}(\lambda_{p,n}^\sharp)$  converges to the price of the Bermudean option when  $p$  goes to infinity.*

*Proof.* Let  $\hat{U}_k$  be the price at time  $-t_k$  of the Bermudean option. The sequence  $(\hat{U}_k)_k$  is a supermartingale admitting the Doob-Meyer decomposition  $\hat{U}_k = \hat{U}_0 + \hat{M}_k - \hat{A}_k$  where  $\hat{M}$  is a square integrable  $(\mathcal{G}_k)_k$ -martingale and  $\hat{A}$  a predictable increasing process for the filtration  $\mathcal{G}$ . The price at time  $-0$  of the Bermudean option also writes

$$\hat{U}_0 = \inf_{X \in L_0^2(\Omega, \mathcal{G}_n, \mathbb{P})} \mathbb{E} \left[ \max_{0 \leq k \leq n} (Z_{t_k} - \mathbb{E}[X | \mathcal{G}_k]) \right].$$

Clearly,  $C_{p,n}(\lambda) \in L^2(\Omega, \mathcal{G}_n, \mathbb{P})$  for any  $\lambda$  such that  $\lambda_0 = 0$  and moreover  $V_{p,n}(\lambda_{p,n}^\#) \geq \hat{U}_0$ . Then, by reproducing the steps in (25), we get

$$0 \leq V_{p,n}(\lambda_{p,n}^\#) - \hat{U}_0 \leq 2 \left\| \hat{M}_n - C_{p,n}(\hat{M}_n) \right\|_2.$$

We deduce from Remark 2.5 that this upper-bound goes to zero as  $p$  tends to infinity.  $\blacksquare$

Most convex optimization algorithms mainly rely on the gradient of the cost function. We end this section by proving that  $V_{p,n}$  is almost everywhere differentiable, which implies that  $\nabla V_{p,n}(\lambda_{p,n}^\#) = 0$ .

**Proposition 3.5.** *Let  $p \geq 1$ . Assume that*

$$\begin{aligned} \forall 1 \leq r \leq k \leq n, \forall F \text{ } \mathcal{F}_{t_k} - \text{measurable}, F \in \mathcal{C}_{p-1,n}, F \neq 0, \exists q \in \{1, \dots, d\} \text{ s.t.} \\ \mathbb{P}(\forall t \in ]t_{r-1}, t_r], D_t^q Z_{t_k} + F = 0 \mid Z_{t_k} > 0) = 0. \end{aligned} \quad (26)$$

*Then, the function  $V_{p,n}$  is differentiable at all points  $\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}}$  with no zero component and its gradient  $\nabla V_{p,n}$  is given by*

$$\nabla V_{p,n}(\lambda) = \mathbb{E} \left[ \mathbb{E} \left[ \hat{H}^{\otimes d}(G_1, \dots, G_n) \mid \mathcal{F}_{t_i} \right]_{\{i\}=\mathcal{I}(\lambda, Z, G)} \right].$$

We refer the reader to section 5.1 for a detailed discussion on which kinds of models and payoffs satisfy (26).

*Proof.* We already know that the function  $V_{p,n}$  is convex. Moreover, for all  $Z$  and  $G$ , the function  $\lambda \mapsto \max_{k \leq n} v_{p,n}(\lambda, k, Z, G)$  has a subdifferential given by

$$\left\{ \sum_{i \in \mathcal{I}(\lambda, Z, G)} \beta_i \mathbb{E}[\hat{H}^{\otimes d}(G_1, \dots, G_n) \mid \mathcal{F}_{t_i}] : \beta_i \geq 0, \beta_i \text{ } \mathcal{F}_T - \text{measurable s.t.} \sum_{i \in \mathcal{I}(\lambda, Z, G)} \beta_i = 1 \right\}$$

Then, the expression of the subdifferential  $\partial V_{p,n}(\lambda)$  ensues from Bertsekas [1973].

$$\partial V_{p,n}(\lambda) = \left\{ \mathbb{E} \left[ \sum_{i \in \mathcal{I}(\lambda, Z, G)} \beta_i \mathbb{E}[\hat{H}^{\otimes d}(G_1, \dots, G_n) \mid \mathcal{F}_{t_i}] \right] : \beta_i \geq 0, \beta_i \text{ } \mathcal{F}_T - \text{meas.}, \sum_i \beta_i = 1 \right\}.$$

It is sufficient to prove for any  $\lambda$  with no zero component, the set  $\mathcal{I}(\lambda, Z, G)$  is almost surely reduced to a single value as in this case the subdifferential  $\partial V_{p,n}(\lambda)$  contains a unique element, which is then the gradient.

By the equality

$$\{\exists t_i \neq t_k ; v_{p,n}(\lambda, i, Z, G) = v_{p,n}(\lambda, k, Z, G)\} = \bigcup_{i < k \leq n} \{v_{p,n}(\lambda, i, Z, G) = v_{p,n}(\lambda, k, Z, G)\},$$

it is sufficient to prove that for any  $i < k \leq n$ ,  $\mathbb{P}(v_{p,n}(\lambda, i, Z, G) = v_{p,n}(\lambda, k, Z, G)) = 0$ . Fix  $i < k$  and set  $X_\lambda = v_{p,n}(\lambda, k, Z, G) - v_{p,n}(\lambda, i, Z, G)$ . According to [Nualart, 1998, Theorem 2.1.3], it is sufficient to prove that  $\|DX_\lambda\|_{L^2([0, T])} > 0$  .a.s to ensure that  $X_\lambda$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$  and hence is almost surely non zero. Since  $\|DX_\lambda\|_{L^2([0, T])}^2 = \int_0^T |D_t X_\lambda|^2 dt \geq \int_{t_i}^{t_k} |D_t X_\lambda|^2 dt \geq \int_{t_i}^{t_k} (D_t^q X_\lambda)^2 dt$  for any  $q \in \{1, \dots, d\}$ .

For  $t \in [0, T]$ , and  $1 \leq q \leq d$ , the Malliavin derivative of  $X_\lambda$  is given by

$$\begin{aligned} D_t^q X_\lambda &= D_t^q(Z_{t_k} - Z_{t_i}) - D_t^q \left( \sum_{\alpha \in A_{p;n}^{\otimes d,k}} \lambda_\alpha \hat{H}_\alpha^{\otimes d}(G_1, \dots, G_n) - \sum_{\alpha \in A_{p;n}^{\otimes d,i}} \lambda_\alpha \hat{H}_\alpha^{\otimes d}(G_1, \dots, G_n) \right) \\ &= D_t^q(Z_{t_k} - Z_{t_i}) - D_t^q \left( \sum_{\alpha \in A_{p;n}^{\otimes d,i:k}} \lambda_\alpha \hat{H}_\alpha^{\otimes d}(G_1, \dots, G_n) \right). \end{aligned}$$

Clearly, w.p.1.  $D_t^q X_\lambda = 0$  for all  $t > t_k$ . Hence,

$$\{D_t^q X_\lambda = 0 \ \forall t \in [0, T] \text{ a.e.}\} \subset \bigcap_{i < r \leq k} \{D_t^q X_\lambda = 0 \ \forall t \in [t_{r-1}, t_r] \text{ a.e.}\}.$$

From Proposition 2.3, we can deduce that for  $i < r \leq k$ , and  $t \in ]t_{r-1}, t_r]$

$$D_t^q X_\lambda = D_t^q(Z_{t_k}) + \frac{1}{\sqrt{h}} \sum_{\alpha \in A_{p;n}^{\otimes d,i:k}, \alpha_r^q \geq 1} \lambda_\alpha \hat{H}_{\alpha - \mathbb{I}(r,q)}^{\otimes d}(G_1, \dots, G_n).$$

Using the locality of the operator  $D$ , we know that a.s  $D_t^q(\phi(S_{t_k})) = 0$  for all  $t \in ]t_{r-1}, t_r]$  on the set  $\{\phi(S_{t_k}) = 0\}$ . Hence, we can write

$$\begin{aligned} \mathbb{P}(\forall t \in ]t_{r-1}, t_r], D_t X_\lambda = 0) &= \mathbb{P} \left( \frac{1}{\sqrt{h}} \sum_{\alpha \in A_{p;n}^{\otimes d,i:k}, \alpha_r^q \geq 1} \lambda_\alpha \hat{H}_{\alpha - \mathbb{I}(r,q)}^{\otimes d}(G_1, \dots, G_n) = 0, \phi(S_{t_k}) = 0 \right) \\ &\quad + \mathbb{P}(\forall t \in ]t_{r-1}, t_r], D_t X_\lambda = 0 \mid Z_{t_k} > 0) \mathbb{P}(Z_{t_k} > 0). \end{aligned}$$

As all the components of  $\lambda$  are non zero,  $\frac{1}{\sqrt{h}} \sum_{\alpha \in A_{p;n}^{\otimes d,i:k}, \alpha_r^q \geq 1} \lambda_\alpha \hat{H}_{\alpha - \mathbb{I}(r,q)}^{\otimes d}(G_1, \dots, G_n)$  is either a non zero constant if  $p = 1$  or it has an absolutely continuous density thanks to Remark 2.7. In both cases, it is a non zero element of  $\mathcal{C}_{p-1,n}$  and

$$\mathbb{P} \left( \frac{1}{\sqrt{h}} \sum_{\alpha \in A_{p;n}^{\otimes d,i:k}, \alpha_r^q \geq 1} \lambda_\alpha \hat{H}_{\alpha - \mathbb{I}(r,q)}^{\otimes d}(G_1, \dots, G_n) = 0 \right) = 0.$$

To treat the other term, we pick a  $q \in \{1, \dots, d\}$  as in the assumption of the proposition (see (26)) and it yields that

$$\mathbb{P}(\forall t \in ]t_{r-1}, t_r], D_t^q X_\lambda = 0 \mid Z_{t_k} > 0) = 0.$$

Hence, we deduce that

$$\|DX_\lambda\|_{L^2([0,T])}^2 \geq \int_{t_{r-1}}^{t_r} (D_t^q X_\lambda)^2 dt > 0 \text{ a.s.}$$

which concludes the proof. ■

### 3.2 The Sample Average Approximation point of view

From (25), we can approximate  $U_0$  by solving the minimization problem (22), which admits at least one solution  $\lambda_{p,n}^\sharp$ , ie.

$$V_{p,n}(\lambda_{p,n}^\sharp) = \inf_{\lambda \in A_{p,n}^{\otimes d}, \lambda_0=0} V_{p,n}(\lambda)$$

where  $V_{p,n}$  defined by (21) is an expectation, which is barely tractable. To practically solve such a problem, two differently approaches are commonly used. Either, one uses a stochastic algorithm or one replaces the expectation by a sample average approximation. In this work, we target large problems, which puts scalability as a primary requirement. The intrinsic sequential nature of stochastic algorithms has led us to prefer the sample average approximation approach. Moreover, we are more interested in the value function at the minimum rather than in its minimizer and unlike stochastic algorithm, standard optimization algorithms provide both at once.

We introduce the sample average approximation of  $V_{p,n}$  defined by

$$V_{p,n}^m(\lambda) = \frac{1}{m} \sum_{i=1}^m \max_{0 \leq k \leq n} v_{p,n}(\lambda, k; Z^{(i)}, G^{(i)})$$

where  $(Z^{(i)}, G^{(i)})_{1 \leq i \leq m}$  are i.i.d samples from the distribution of  $(Z, G)$ .

For large enough  $m$ ,  $V_{p,n}^m$  inherits from the smoothness of  $V_{p,n}$  and is in particular convex and a.s. differentiable at any point with no zero component. Then, we easily deduce from Proposition 3.1 that there exists  $\lambda_{p,n}^m$  such that

$$V_{p,n}^m(\lambda_{p,n}^m) = \inf_{\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}}, \lambda_0=0} V_{p,n}^m(\lambda)$$

and moreover  $\nabla V_{p,n}^m(\lambda_{p,n}^m) = 0$ . The main difficulty in studying the convergence of  $V_{p,n}^m(\lambda_{p,n}^m)$  when  $m$  goes to infinity comes from the non compactness of the set  $\mathbb{R}^{A_{p,n}^{\otimes d}}$ . To circumvent this problem, we adapt to non strictly convex problems the technique used in Jourdain and Lelong [2009].

**Proposition 3.6.** *The sequence  $V_{p,n}^m(\lambda_{p,n}^m)$  converges a.s. to  $V_{p,n}(\lambda_{p,n}^\sharp)$  when  $m \rightarrow \infty$ . Moreover, the distance between  $\lambda_{p,n}^m$  and the convex set of minimizers in (22) converges to zero as  $m$  goes to infinity.*

*Proof.* The random function  $\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}} \mapsto \max_{0 \leq k \leq n} v_{p,n}(\lambda, k; Z, G)$  is a.s. continuous. For

$\Lambda > 0$ ,

$$\begin{aligned}
& \sup_{|\lambda| \leq \Lambda} \max_{0 \leq k \leq n} v_{p,n}(\lambda, k; Z, G) \\
& \leq \max_{0 \leq k \leq n} Z_{t_k} + \sup_{|\lambda| \leq \Lambda} \max_{0 \leq k \leq n} \sum_{\alpha \in A_{p,n}^{\otimes d}} \lambda_{\alpha} \mathbb{E} \left[ \hat{H}_{\alpha}^{\otimes d}(G_1, \dots, G_n) \middle| \mathcal{F}_{t_k} \right] \\
& \leq \max_{0 \leq k \leq n} Z_{t_k} + \Lambda \sup_{|\lambda|=1} \max_{0 \leq k \leq n} \sum_{\alpha \in A_{p,n}^{\otimes d}} \lambda_{\alpha} \mathbb{E} \left[ \hat{H}_{\alpha}^{\otimes d}(G_1, \dots, G_n) \middle| \mathcal{F}_{t_k} \right] \\
& \leq \max_{0 \leq k \leq n} Z_{t_k} + \Lambda \max_{0 \leq k \leq n} \sum_{\alpha \in A_{p,n}^{\otimes d}} \left| \mathbb{E} \left[ \hat{H}_{\alpha}^{\otimes d}(G_1, \dots, G_n) \middle| \mathcal{F}_{t_k} \right] \right| \\
& \leq \max_{0 \leq k \leq n} Z_{t_k} + \Lambda \sum_{\alpha \in A_{p,n}^{\otimes d}} \max_{0 \leq k \leq n} \mathbb{E} \left[ \left| \hat{H}_{\alpha}^{\otimes d}(G_1, \dots, G_n) \right| \middle| \mathcal{F}_{t_k} \right] \\
& \leq \max_{0 \leq k \leq n} Z_{t_k} + \Lambda \sum_{\alpha \in A_{p,n}^{\otimes d}} \sum_{k=0}^n \mathbb{E} \left[ \left| \hat{H}_{\alpha}^{\otimes d}(G_1, \dots, G_n) \right| \middle| \mathcal{F}_{t_k} \right].
\end{aligned}$$

The right hand side of the above inequality is integrable. We apply [Rubinstein and Shapiro, 1993, Lemma A1 Chapter 2] to deduce that a.s.  $V_{p,n}^m$  converges locally uniformly to  $V_{p,n}$ . From the proof of the Proposition 3.1, there exists  $\Lambda > 0$  such that

$$\gamma = \inf_{|\lambda - \lambda_{p,n}^{\#}| \geq \Lambda} V_{p,n}(\lambda) - V_{p,n}(\lambda_{p,n}^{\#}) > 0.$$

The local uniform convergence of  $V_{p,n}^m$  to  $V_{p,n}$  ensures that

$$\exists m_{\gamma} \in \mathbb{N}^*, \forall m \geq m_{\gamma}, \forall \lambda \text{ s.t. } |\lambda - \lambda_{p,n}^{\#}| \leq \Lambda, \quad \left| V_{p,n}^m(\lambda) - V_{p,n}(\lambda) \right| \leq \frac{\gamma}{3}.$$

For  $m \geq m_{\gamma}$  and  $\lambda$  such that  $|\lambda - \lambda_{p,n}^{\#}| \geq \Lambda$ , we deduce, using the convexity of  $V_{p,n}^m$ , that

$$\begin{aligned}
& V_{p,n}^m(\lambda) - V_{p,n}^m(\lambda_{p,n}^{\#}) \\
& \geq \frac{|\lambda - \lambda_{p,n}^{\#}|}{\Lambda} \left\{ V_{p,n}^m \left( \lambda_{p,n}^{\#} + \Lambda \frac{\lambda - \lambda_{p,n}^{\#}}{|\lambda - \lambda_{p,n}^{\#}|} \right) - V_{p,n}^m(\lambda_{p,n}^{\#}) \right\} \\
& \geq \frac{|\lambda - \lambda_{p,n}^{\#}|}{\Lambda} \left\{ V_{p,n} \left( \lambda_{p,n}^{\#} + \Lambda \frac{\lambda - \lambda_{p,n}^{\#}}{|\lambda - \lambda_{p,n}^{\#}|} \right) - V_{p,n}(\lambda_{p,n}^{\#}) - \frac{2\gamma}{3} \right\} \geq \frac{\gamma}{3}.
\end{aligned}$$

Since  $V_{p,n}^m(\lambda_{p,n}^m) - V_{p,n}^m(\lambda_{p,n}^{\#}) \leq 0$ , we conclude that the above inequality does not hold for  $\lambda_{p,n}^m$ , which proves that  $|\lambda_{p,n}^m - \lambda_{p,n}^{\#}| < \Lambda$  for  $m \geq m_{\gamma}$ .

Hence, for  $m \geq m_{\gamma}$ , it is sufficient to minimize  $V_{p,n}^m$  on the compact set  $\{\lambda : |\lambda - \lambda_{p,n}^{\#}| \leq \Lambda\}$ . Now, we can apply [Rubinstein and Shapiro, 1993, Theorem A1 of Chapter 2] to prove that  $V_{p,n}^m(\lambda_{p,n}^m)$  converges to  $V_{p,n}(\lambda_{p,n}^{\#})$  a.s. when  $m$  goes to infinity. The second assertion of our proposition is discussed right after the proof of Theorem A1 in Rubinstein and Shapiro [1993].  $\blacksquare$

Although  $V_{p,n}^m$  is not twice differentiable and the classical central limit theorem for sample average approximations cannot be applied, we can study the variance of  $V_{p,n}^m(\lambda_{p,n}^m)$  and obtain

some asymptotic bounds. Before stating our result, we introduce, for  $\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}}$ , the notation  $M_k(\lambda) = \mathbb{E}[C_{p,n}(\lambda) | \mathcal{F}_{t_k}]$  for  $0 \leq k \leq n$ . We write  $M_k^{(i)}(\lambda)$  for the value computed using the sample  $G^{(i)}$ .

**Proposition 3.7.** *Assume  $\lambda_{p,n}^\#$  is unique. Then,*

$$\frac{1}{m} \sum_{i=1}^m \left( \max_{0 \leq k \leq n} Z_{t_k}^{(i)} - M_k^{(i)}(\lambda_{p,n}^m) \right)^2 - V_{p,n}^m(\lambda_{p,n}^m)^2$$

*is a convergent estimator of  $\text{Var}(\max_{k \leq 0 \leq n} Z_{t_k} - M_k(\lambda_{p,n}^\#))$  and moreover if  $\lambda_{p,n}^m$  is bounded,  $\lim_{m \rightarrow \infty} m \text{Var} \left( V_{p,n}^m(\lambda_{p,n}^m) \right) = \text{Var}(\max_{k \leq 0 \leq n} Z_{t_k} - M_k(\lambda_{p,n}^\#))$ .*

*Proof.* We know that  $V_{p,n}^m(\lambda_{p,n}^m)$  converges a.s. to  $V_{p,n}(\lambda_{p,n}^\#)$ . Following the beginning of the proof of Proposition 3.6, one can easily prove that a.s. the sequence of random functions  $\zeta^m : \lambda \mapsto \zeta^m(\lambda) = \frac{1}{m} \sum_{i=1}^m \left( \max_{0 \leq k \leq n} Z_{t_k}^{(i)} - M_k^{(i)}(\lambda) \right)^2$  converges locally uniformly to the function  $\lambda \mapsto \mathbb{E}[(\max_{0 \leq k \leq n} Z_{t_k} - M_k(\lambda))^2]$ . We have already seen that for large enough  $m$ , we can assume to have solved the optimization problem under a compact constraint. Hence, we deduce that  $\frac{1}{m} \sum_{i=1}^m \left( \max_{0 \leq k \leq n} Z_{t_k}^{(i)} - M_k^{(i)}(\lambda_{p,n}^m) \right)^2$  converges a.s. to  $\mathbb{E}[(\max_{0 \leq k \leq n} Z_{t_k} - M_k(\lambda_{p,n}^\#))^2]$ . This proves the first statement of the proposition.

As  $\text{Var} \left( V_{p,n}^m(\lambda_{p,n}^\#) \right) = m^{-1} \text{Var}(\max_{k \leq 0 \leq n} Z_{t_k} - M_k(\lambda_{p,n}^\#))$ , it is sufficient to compute

$$\begin{aligned} & \mathbb{E} \left[ \left( V_{p,n}^m(\lambda_{p,n}^m) - V_{p,n}^m(\lambda_{p,n}^\#) \right)^2 \right] \\ & \leq \frac{1}{m} \sum_{i=1}^m \mathbb{E} \left[ \max_k \left| M_k^{(i)}(\lambda_{p,n}^m) - M_k^{(i)}(\lambda_{p,n}^\#) \right|^2 \right] \\ & \leq \frac{1}{m} \sum_{i=1}^m \mathbb{E} \left[ \left| \lambda_{p,n}^m - \lambda_{p,n}^\# \right|^2 \max_k \left| \mathbb{E} \left[ \hat{H}^{\otimes d}(G_1^{(i)}, \dots, G_n^{(i)}) \mid \mathcal{F}_{t_k} \right] \right|^2 \right] \\ & \leq \frac{16}{9} \mathbb{E} \left[ \left| \lambda_{p,n}^m - \lambda_{p,n}^\# \right|^4 \right]^{1/2} \mathbb{E} \left[ \left| \hat{H}^{\otimes d}(G_1^{(i)}, \dots, G_n^{(i)}) \right|^4 \right]^{1/2} \end{aligned}$$

where we have used Cauchy Schwartz' inequality and Doob's maximal inequality. Then, we easily conclude that  $V_{p,n}^m(\lambda_{p,n}^m) - V_{p,n}^m(\lambda_{p,n}^\#)$  converges to 0 in  $\mathbb{L}^2$  if  $\lambda_{p,n}^m$  is bounded. Hence,  $\lim_{m \rightarrow \infty} \text{Var} \left( V_{p,n}^m(\lambda_{p,n}^m) \right) - \text{Var} \left( V_{p,n}^m(\lambda_{p,n}^\#) \right) = 0$ .  $\blacksquare$

Proposition 3.7 enables us to monitor the variance of our estimator online as for a standard Monte Carlo estimator. Even though the terms involved in  $V_{p,n}^m(\lambda_{p,n}^m)$  are not independent, the classical variance estimator gives the right result. In practice, one should not feel concerned with the boundedness condition used in the proposition as we know from the proof of Proposition 3.6 that for large enough  $m$  we can impose a compactness constraint to the optimization problem without changing its result. Hence, one can pragmatically rely on the proposed variance estimator.

## 4 The algorithm

Any optimization algorithm requires to repeatedly compute  $V_{p,n}^m$  and therefore the truncated chaos expansion, which becomes the most time consuming part of our approach as the dimension and/or  $p$  increase. A lot of computational time can be saved by considering slightly modified martingales, which only start the first time the option goes in the money.

## 4.1 An improved set of martingales

We define the first time the option goes in the money by

$$\tau_0 = \inf\{k \geq 0 : Z_{t_k} > 0\} \wedge n,$$

which is a  $\mathcal{F}$ -stopping time and becomes a  $\mathcal{G}$ -stopping time when the sequence  $(Z_{t_k})_k$  is  $\mathcal{G}$ -adapted. To consider martingales only starting once the option has been in the money, we define

$$N_k(\lambda) = \sum_{\ell=1}^k (M_\ell(\lambda) - M_{\ell-1}(\lambda)) \mathbf{1}_{\ell-1 \geq \tau_0} = (M_k(\lambda) - M_{\tau_0}(\lambda)) \mathbf{1}_{k > \tau_0} = M_k(\lambda) - M_{k \wedge \tau_0}(\lambda)$$

We easily check that  $N(\lambda)$  is a  $(\mathcal{F}_{t_k})_{0 \leq k \leq n}$ -martingale. It is clear from the proof proposed by Rogers [2002] that in the dual price of a Bermudan option (see (3)) the maximum can be shrunk to the random interval  $[\tau_0, n]$ . Hence, it is sufficient to consider

$$\inf_{\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}}, \lambda_0=0} \mathbb{E} \left[ \max_{\tau_0 \leq k \leq n} (Z_{t_k} - M_k(\lambda)) \right].$$

Using Doob's stopping theorem, we have, for any fixed  $\lambda$ ,

$$\mathbb{E} \left[ \max_{\tau_0 \leq k \leq n} (Z_{t_k} - M_k(\lambda)) \right] = \mathbb{E} \left[ \max_{\tau_0 \leq k \leq n} (Z_{t_k} - (M_k(\lambda) - M_{\tau_0}(\lambda))) \right] = \mathbb{E} \left[ \max_{\tau_0 \leq k \leq n} (Z_{t_k} - N_k(\lambda)) \right].$$

We deduce from this equality that minimizing over either set of martingales  $M(\lambda)$  or  $N(\lambda)$  leads to the same minimum value and that both problems share the same properties, which justifies why we did not take into account the in-the-money condition for the theoretical study. However, considering the set of martingales  $N^\lambda$  is far more efficient from a practical point of view.

In our numerical examples, we modify  $V_{p,n}$  and  $V_{p,n}^m$  to take into account this improvement and consider instead

$$\tilde{V}_{p,n}(\lambda) = \mathbb{E} \left[ \max_{\tau_0 \leq k \leq n} (Z_{t_k} - N_k(\lambda)) \right] \quad \text{and} \quad \tilde{V}_{p,n}^m(\lambda) = \frac{1}{m} \sum_{i=1}^m \max_{\tau_0 \leq k \leq n} (Z_{t_k}^{(i)} - N_k^{(i)}(\lambda)).$$

The idea of using martingales starting from the first time the option goes in the money is actually owed to Rogers [2002]. Although he did not discuss it much, this was his choice in the examples he treated.

## 4.2 Our implementation of the algorithm

To practically compute the infimum of  $\tilde{V}_{p,n}^m$ , we advise to use a gradient descent algorithm, see Algorithm 1. The efficiency of such an approach mainly depends on the computation of the descend direction. When the problem is not twice differentiable, the gradient at the current point is used as a descent direction but it often needs to be scaled, which makes the choice of the step size  $\alpha_\ell$  a burning issue to ensure a fast numerical convergence. We refer to Boyd et al. [2003] for a comprehensive survey of several step size rules. After many tests, we found that the step size rule proposed by Polyak [1987] was the best performing one in our context

$$\alpha_\ell = \frac{\tilde{V}_{p,n}^m(x_\ell) - v^\#}{\|\nabla \tilde{V}_{p,n}^m(x_\ell)\|^2}$$



where  $v^\sharp$  is the price of the American option we are looking for. In practice, we use the price of the associated European option instead of  $v^\sharp$ , which makes  $\alpha_\ell$  too large and explains the need of the magnitude factor  $\gamma$ . The value of the European price does not need to be very accurate. A decent and fast approximation can be computed with a few thousand samples within few seconds no matter the dimension of the problem.

```

1 Generate  $(G^{(1)}, Z^{(1)}), \dots, (G^{(m)}, Z^{(m)})$   $m$  i.i.d. samples following the law of  $(Z, G)$  ;
2  $x_0 \leftarrow 0 \in \mathbb{R}^{A_{p,n}^{\otimes d}}$ ;
3  $\ell \leftarrow 0, \gamma \leftarrow 1, d_0 \leftarrow 0, v_0 \leftarrow \infty$  ;
4 while True do
5   Compute  $v_{\ell+1/2} \leftarrow \tilde{V}_{p,n}^m(x_\ell - \gamma \alpha_\ell d_\ell)$  ;
6   if  $v_{\ell+1/2} < v_\ell$  then
7      $x_{\ell+1} \leftarrow x_\ell - \gamma \alpha_\ell d_\ell$  ;
8      $v_{\ell+1} \leftarrow v_{\ell+1/2}$  ;
9      $d_{\ell+1} \leftarrow \nabla \tilde{V}_{p,n}^m(x_{\ell+1})$  ;
10    if  $\frac{|v_{\ell+1} - v_\ell|}{v_\ell} \leq \varepsilon$  then return
11  else
12     $\gamma \leftarrow \gamma/2$  ;
13  end
14 end

```

**Algorithm 1:** Sample Average Approximation of the dual price

To better understand how this algorithm works, it is important to note that as  $N(\lambda)$  linearly depends on  $\lambda$ ,  $N(\lambda) = \lambda \cdot \nabla_\lambda N(\lambda)$  and therefore both the value function and its gradient are computed at the same time without extra cost. So,  $\nabla \tilde{V}_{p,n}^m(x_{\ell+1})$  is not actually computed on line 9 but at the same time as  $v_{\ell+1/2}$  on line 5.

**The HPC approach.** Our method targets large problems with as many as several thousands of components for  $\lambda$ . This requires to design a scalable algorithm capable of making the most of cluster architectures with hundreds of nodes. At each iteration, the computation of  $\tilde{V}_{p,n}^m$  and  $\nabla \tilde{V}_{p,n}^m$  is nothing but a standard Monte Carlo method and it inherits from its embarrassingly parallel nature.

A parallel algorithm for distributed memory systems based on the master/slave paradigm is proposed in Algorithm 2. At the beginning, each process samples a bunch of the  $m$  paths (lines 1–3). Then, at each iteration the master process broadcasts the value of  $d_\ell$ ,  $x_\ell$ ,  $\alpha_\ell$  and  $\gamma$  (line 7 of Algorithm 1). With these new values, each process computes its contribution to  $\tilde{V}_{p,n}^m(x_\ell - \gamma \alpha_\ell d_\ell)$  and  $\nabla \tilde{V}_{p,n}^m(x_\ell - \gamma \alpha_\ell d_\ell)$  (lines 8–9) and the Monte Carlo summations are obtained by two simple reductions (line 11). Then, the master process tests whether the move is admissible and updates the parameter for the next iteration or returns the solution if the algorithm is not moving enough anymore. This part carried out by the master process is very fast compared to the rest of the code and we dare say that there is no centralized computation in our algorithm. Moreover the communications are reduced to four broadcasts, which guarantees an almost perfect very good scalability. The number of communications is monitored by the number of function evaluations, which remains quite small (between 10 and 20). We study the efficiency of our algorithm on a few examples at the end of Section 5.

```

1 In parallel do
2   | Generate  $(G^{(1)}, Z^{(1)}), \dots, (G^{(m)}, Z^{(m)})$   $m$  i.i.d. samples following the law of  $(Z, G)$ 
3 end
4  $x_0 \leftarrow 0 \in \mathbb{R}^{A_{p,n}^{\otimes d}}$ ;
5  $\ell \leftarrow 0, \gamma \leftarrow 1, d_0 \leftarrow 0, v_0 \leftarrow \infty$ ;
6 while True do
7   | Broadcast  $x_\ell, d_\ell, \gamma, \alpha_\ell$ ;
8   | In parallel do
9     | Compute  $\max_{\tau_0 \leq k \leq n} (Z_{t_k}^{(i)} - N_k^{(i)}(x_\ell - \gamma \alpha_\ell d_\ell))$  for  $i = 1, \dots, m$ 
10  | end
11  | Make a reduction of the above contributions to obtain  $\tilde{V}_{p,n}^m(x_\ell - \gamma \alpha_\ell d_\ell)$  and
    |  $\nabla \tilde{V}_{p,n}^m(x_\ell - \gamma \alpha_\ell d_\ell)$ ;
12  |  $v_{\ell+1/2} \leftarrow \tilde{V}_{p,n}^m(x_\ell - \gamma \alpha_\ell d_\ell)$ ;
13  | if  $v_{\ell+1/2} < v_\ell$  then
14    |  $x_{\ell+1} \leftarrow x_\ell - \gamma \alpha_\ell d_\ell$ ;
15    |  $v_{\ell+1} \leftarrow v_{\ell+1/2}$ ;
16    |  $d_{\ell+1} \leftarrow \nabla \tilde{V}_{p,n}^m(x_{\ell+1})$ ;
17    | if  $\frac{|v_{\ell+1} - v_\ell|}{v_\ell} \leq \varepsilon$  then return
18  | else
19    |  $\gamma \leftarrow \gamma/2$ ;
20  | end
21 end

```

**Algorithm 2:** Parallel implementation of the Sample Average Approximation of the dual price

**Study of the complexity.** Most of the computational time is spent computing the martingale part; remember that the cardinality of  $\mathcal{C}_{p,n}$  is given by  $\binom{nd+p}{nd} = \frac{(nd+p)\dots(nd+1)}{p!}$ . Using martingales only starting once the option has been in the money enables us to only compute the martingale part on paths going in the money strictly before maturity time. Depending on the product, this may allow for saving a lot of computational time. The complexity of one iteration of loop line 3 in Algorithm 1 is proportional to

$$\#\{\text{paths in the money strictly before } T\} \times \binom{nd+p}{nd}.$$

The payoffs are computed once and for all before starting the descent algorithm. It is worth noting that its computational cost becomes negligible compared to the optimization part when the dimension of the model or the number of dates increase, the most demanding computation being the evaluation of the martingale decomposition.

## 5 Applications

### 5.1 Some frameworks satisfying the assumption of Proposition 3.5

Let  $(r_t)_t$  be the instantaneous interest rate supposed to be deterministic.

### 5.1.1 A put basket option in the multi-dimensional Black Scholes model

The  $d$ -dimensional Black Scholes model writes for  $j \in \{1, \dots, d\}$

$$dS_t^j = S_t^j((r_t - \delta^j)dt + \sigma^j L_j dB_t)$$

where  $W$  is a Brownian motion with values in  $\mathbb{R}^d$ ,  $\sigma_t = (\sigma_t^1, \dots, \sigma_t^d)$  is the vector of volatilities, assumed to be deterministic and positive at all times,  $\delta = (\delta^1, \dots, \delta^d)$  is the vector of instantaneous dividend rates and  $L_j$  is the  $j$ -th row of the matrix  $L$  defined as a square root of the correlation matrix  $\Gamma$ , ie.  $\Gamma = LL'$ . Moreover, we assume that  $L$  is lower triangular. Clearly, for every  $t$ , the random vector  $S_t$  is an element of  $\mathbb{D}^{1,2}$ .

The payoff of the put basket option writes as  $\phi(S_t) = \left(K - \sum_{i=1}^d \omega^i S_t^i\right)_+$  where  $\omega = (\omega^1, \dots, \omega^d)$  is a vector of real valued weights. The function  $\phi$  is Lipschitz continuous and hence  $\phi(S_t) \in \mathbb{D}^{1,2}$  for all  $t$ . Moreover, for  $s \leq t$  and  $q \in \{1, \dots, d\}$ , we have on the set  $\{\phi(S_t) > 0\}$

$$D_s^q \phi(S_t) = \sum_{j=1}^d \omega^j S_t^j \sigma^j L_{j,q}.$$

In particular for  $q = d$ , we get  $D_s^d \phi(S_t) = \omega^d S_t^d \sigma^d L_{d,d}$ .

Let  $1 \leq k \leq n$  and  $F$  be a non zero and  $\mathcal{F}_{t_k}$ -measurable element of  $\mathcal{C}_{p-1,n}$ , ie.

$$F = \sum_{\alpha \in A_{p-1,n}^{\otimes d,k}} \lambda_\alpha \hat{H}_\alpha^{\otimes d}(G_1, \dots, G_n)$$

for some  $\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}}$ . Let  $1 \leq r \leq k$ .

$$\begin{aligned} & \mathbb{P}\left(\forall t \in ]t_{r-1}, t_r], D_t^d \phi(S_{t_k}) + F = 0 \mid \phi(S_{t_k}) > 0\right) \\ &= \mathbb{P}\left(\forall t \in ]t_{r-1}, t_r], \omega^d S_{t_k}^d \sigma_t^d L_{d,d} + F = 0 \mid \phi(S_{t_k}) > 0\right) \\ &\leq \frac{\mathbb{P}\left(\forall t \in ]t_{r-1}, t_r], \omega^d S_{t_k}^d \sigma_t^d L_{d,d} + F = 0\right)}{\mathbb{P}(\phi(S_{t_k}) > 0)}. \end{aligned} \quad (27)$$

If  $p = 1$ , then  $F$  is a deterministic non zero constant. In this case, the numerator vanishes because  $S_{t_k}^d$  has a density. Assume  $p \geq 2$ , then  $F$  is a multivariate polynomial with global degree  $p-1 \geq 1$ . Then we can find  $\ell \in \{1, \dots, k\}$ ,  $q \in \{1, \dots, d\}$  and  $\alpha$  such that  $\alpha_\ell^q \geq 1$  and  $\lambda_\alpha \neq 0$ . Let  $\hat{\mathcal{G}}$  be the sigma algebra generated by  $(G_i^j, 1 \leq i \leq k, 1 \leq j \leq d, (i, j) \neq (\ell, q))$ .

$$\mathbb{P}\left(\forall t \in ]t_{r-1}, t_r], \omega^d S_t^d \sigma_t^d L_{d,d} + F = 0\right) = \mathbb{E}\left[\mathbb{P}\left(\forall t \in ]t_{r-1}, t_r], \omega^d S_t^d \sigma_t^d L_{d,d} + F = 0 \mid \hat{\mathcal{G}}\right)\right].$$

Conditioning on  $\hat{\mathcal{G}}$ , the random variable  $\omega^d S_t^d \sigma_t^d L_{d,d} + F$  only depends on  $G_\ell^q$ . Consider the algebraic equation for  $x \in \mathbb{R}$

$$a e^{bx+c} = P(x) \quad (28)$$

where  $(a, b, c) \in \mathbb{R}^3, a \neq 0, b \neq 0$  and  $P$  is polynomial with degree  $p-1 \geq 1$ . Let  $f(x) = a e^{bx+c} - P(x)$ ,  $f^{(p)}(x) = ab^p e^{bx+c}$ . Clearly,  $f^{(p)}$  never vanishes, which ensures that  $f$  has at most  $p$  different roots. Hence, we deduce that for any  $t \in ]t_{r-1}, t_r]$ ,  $\mathbb{P}\left(\omega^d S_t^d \sigma_t^d L_{d,d} + F = 0 \mid \hat{\mathcal{G}}\right) = 0$ . Combining this result along with (27) proves that Equation (26) holds in this setting.

### 5.1.2 A put option on the minimum of a basket in the multi-dimensional Black Scholes model

We use the notation of the previous example. The payoff of the put option on the minimum of  $d$  assets write  $\phi(S_t) = (K - \min_j(S_t^j))_+$ . One can prove by induction on  $d$  that the function  $x \in \mathbb{R}^d \mapsto \min_j(x^j)$  is 1-Lipschitz for the 1-norm on  $\mathbb{R}^d$ . Hence, as the positive part function is also Lipschitz, the payoff function  $\phi$  is Lipschitz. Then, [Nualart, 1998, Proposition 1.2.4] yields that for all  $t \in [0, T]$ ,  $\phi(S_t) \in \mathbb{D}^{1,2}$  and for all  $q \in \{1, \dots, d\}$ ,

$$D^q(\phi(S_t)) = \sum_{j=1}^d \partial_{x^j} \phi(S_t) D^q(S_t^j) = \sum_{j=1}^d \partial_{x^j} \phi(S_t) S_t^j \sigma^j L_{j,q}.$$

With our choice for the matrix  $L$ ,

$$D^d(\phi(S_t)) = \partial_{x^d} \phi(S_t) S_t^d \sigma^d L_{d,d} = -S_t^d \sigma^d L_{d,d} \mathbf{1}_{\phi(S_t) > 0} \mathbf{1}_{\min_j(S_t^j) = S_t^d}.$$

Let  $1 \leq k \leq n$  and  $F$  be a non zero and  $\mathcal{F}_{t_k}$ -measurable element of  $\mathcal{C}_{p-1,n}$ . For  $1 \leq r \leq k$ ,

$$\begin{aligned} & \mathbb{P} \left( \forall t \in ]t_{r-1}, t_r], D_t^d \phi(S_{t_k}) + F = 0 \mid \phi(S_{t_k}) > 0 \right) \\ &= \mathbb{P} \left( \forall t \in ]t_{r-1}, t_r], -S_t^d \sigma^d L_{d,d} + F = 0 \mid \phi(S_{t_k}) > 0, \min_j(S_t^j) = S_t^d \right) \mathbb{P} \left( \min_j(S_t^j) = S_t^d \right) \\ &+ \mathbb{P} \left( \forall t \in ]t_{r-1}, t_r], F = 0 \mid \phi(S_{t_k}) > 0, \min_j(S_t^j) \neq S_t^d \right) \mathbb{P} \left( \min_j(S_t^j) \neq S_t^d \right) \end{aligned}$$

Clearly, the second term in the above sum is zero as  $F$  has a density. Hence,

$$\mathbb{P} \left( \forall t \in ]t_{r-1}, t_r], D_t^d \phi(S_{t_k}) + F = 0 \mid \phi(S_{t_k}) > 0 \right) \leq \frac{\mathbb{P} \left( \forall t \in ]t_{r-1}, t_r], -S_t^d \sigma^d L_{d,d} + F = 0 \right)}{\mathbb{P}(\phi(S_{t_k}) > 0)}.$$

We conclude as in the case of the put basket option.

### 5.1.3 A put option in the Heston model

The Heston model can be written

$$\begin{aligned} dS_t &= S_t(r_t dt + \sqrt{\sigma_t}(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2)) \\ d\sigma_t &= \kappa(\theta - \sigma_t)dt + \xi \sqrt{\sigma_t} dW_t^1. \end{aligned}$$

For  $s \leq t$ ,  $D_s^2 S_t = S_t \sqrt{1 - \rho^2} \sqrt{\sigma_t}$ . Conditionally on  $W^1$ ,  $D_s^2 S_t$  writes as  $a e^{bW_t^2 + c}$  and we can unfold the same reasoning as after (28).

## 5.2 Numerical experiments

In this part, we present results obtained from a sequential implementation of our approach as described in Algorithm 1. The computations are run on a standard laptop with an Intel Core i5 processor 2.9 Ghz. For each experiment, we report the price obtained using Algorithm 1 along with its computational time and standard deviation.

### 5.2.1 Examples in the Black Scholes models

We consider the  $d$ -dimensional Black Scholes as presented in Section 5.1.1. For the sake of simplicity in choosing the parameters, we have decide to use the same correlation between all the assets, which amounts to considering the following simple structure for  $\Gamma$ .

$$\Gamma = \begin{pmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \dots & \rho & 1 \end{pmatrix} \quad (29)$$

where  $\rho \in ]-1/(d-1), 1]$  to ensure that  $\Gamma$  is positive definite.

**A basket option in the Black–Scholes model.** We consider a put option on several assets as presented in Section 5.1.1. We report in Table 1 the price obtained with our approach for  $m = 20,000$ . The last column *reference price* corresponds to the prices reported in Schoenmakers et al. [2013] on the same examples. These *reference* prices were obtained within a few minutes according to the authors whereas here we manage to get similar values within a few seconds. We can see that a second order chaos expansion,  $p = 2$ , already gives very accurate results within a few tenths of a second for a 5-dimensional problem with 6 dates, which proves the impressive efficiency of our approach.

$p$	$n$	$S_0$	price	Stdev	time (sec.)	reference price
2	3	100	2.27	0.029	0.17	2.17
3	3	100	2.23	0.025	0.9	2.17
2	3	110	0.56	0.014	0.07	0.55
3	3	110	0.53	0.012	0.048	0.55
2	6	100	2.62	0.021	0.91	2.43
3	6	100	2.42	0.021	14	2.43
2	6	110	0.61	0.012	0.33	0.61
3	6	110	0.55	0.008	10	0.61

Table 1: Prices for the put basket option with parameters  $T = 3$ ,  $r = 0.05$ ,  $K = 100$ ,  $\rho = 0$ ,  $\sigma^j = 0.2$ ,  $\delta^j = 0$ ,  $d = 5$ ,  $\omega^j = 1/d$ .

**A call on the maximum of  $d$  assets in the Black–Scholes model.** We consider a call option on the maximum of  $d$  assets in the Black Scholes model. As in the previous example, the last column *reference price* corresponds to the prices reported in Schoenmakers et al. [2013] on the same examples. With no surprise, the computational time increases exponentially with the dimension  $n \times d$  and the degree  $p$ . Whereas a second order expansion provides very accurate results for the basket option, it only gives a rough upper-bound for the call option on the maximum of  $d$  assets. Considering a third order expansion  $p = 3$  takes far longer but enables us to get very tight upper-bounds.

**A geometric basket option in the Black–Scholes model** Benchmarking a new method on high dimensional products becomes hardly feasible as almost no high dimensional American

$d$	$p$	$m$	$S_0$	price	Stdev	time (sec.)	reference price
2	2	20,000	90	10.18	0.07	0.4	8.15
2	3	20,000	90	8.5	0.05	4.1	8.15
2	2	20,000	100	16.2	0.06	0.54	14.01
2	3	20,000	100	14.4	0.06	5.6	14.01
5	2	20,000	90	21.2	0.09	2	16.77
5	3	40,000	90	16.3	0.05	210	16.77
5	2	20,000	100	30.7	0.09	3.4	26.34
5	3	40,000	100	26.0	0.05	207	26.34

Table 2: Prices for the call option on the maximum of  $d$  assets with parameters  $T = 3$ ,  $r = 0.05$ ,  $K = 100$ ,  $\rho = 0$ ,  $\sigma^j = 0.2$ ,  $\delta^j = 0.1$ ,  $n = 9$ .

options can be priced accurately in a reasonable time. An exception to this is the geometric option with payoff  $(K - (\prod_{j=1}^d S_t^j)^{1/d})_+$  for the put option. Easy calculations show that the price of this  $d$ -dimensional option equals the one of the 1-dimensional option with parameters

$$\hat{S}_0 = \left( \prod_{j=1}^d S_0^j \right)^{1/d}; \quad \hat{\sigma} = \frac{1}{d} \sqrt{\sum_{i,j} \sigma^i \sigma^j \Gamma_{ij}}; \quad \hat{\delta} = \frac{1}{d} \sum_{j=1}^d \left( \delta^j + \frac{1}{2} (\sigma^j)^2 \right) - \frac{1}{2} (\hat{\sigma})^2.$$

Table 3 summarizes the correspondence values used in the examples.

$d$	$S_0$	$\sigma$	$\rho$	$\hat{S}_0$	$\hat{\sigma}$	$\hat{\delta}$
2	100	0.2	0	100	0.14	0.01
10	100	0.3	0.1	100	0.131	0.036
40	100	0.3	0.1	100	0.105	0.039

Table 3: Correspondence table for the parameters of the geometric options with  $\delta^j = 0$ .

$d$	$\sigma^j$	$\rho$	$p$	$m$	price	Stdev	time(sec)	1-d price
2	0.2	0	2	5000	4.32	0.04	0.018	4.20
2	0.2	0	3	5000	4.15	0.04	1.3	4.20
10	0.3	0.1	1	5000	5.50	0.06	0.12	4.60
10	0.3	0.1	2	20000	4.55	0.02	17	4.60
40	0.3	0.1	1	10000	4.4	0.03	1.4	3.69
40	0.3	0.1	2	20000	3.61	0.02	170	3.69

Table 4: Prices for the geometric basket put option with parameters  $T = 1$ ,  $r = 0.0488$  (it corresponds to a 5% annual interest rate),  $K = 100$ ,  $\delta^j = 0$ ,  $n = 9$ .

The 1- $d$  price is computed using a tree method with several thousand steps. We can see in Table 4 that a second order approximation gives very accurate result within a few seconds for an option with 10 underlying assets, which proves the efficiency of our approach. We cannot

beat the curse of dimensionality, which slows down of algorithm for very large problems. For an option on 40 assets, we obtain a price up to a 3% relative error within 3 minutes which is already very fast for such a high dimensional problem. The number of terms involved in the chaos expansion can become very large: for  $d = 40$  and  $p = 2$ , there are 65340 elements in  $\mathcal{C}_{p,n}$ . Even though we are not working in a linear algebra framework, it is advisable to ensure that the number of samples  $m$  used in the sample average approximation is larger than the number of free parameters in the optimization problem. When  $m$  becomes too small, we may face an over-fitting phenomenon as the number of parameters is far too large compared to the information contained in the sample average approximation. This probably explains why the price obtained for  $p = 2$ ,  $d = 40$  and  $m = 40$  is slightly smaller than the true price.

In the next paragraph, we test the scalability of Algorithm 2 on this particular examples for a larger number of samples.

### 5.2.2 Scalability of the parallel algorithm

We consider the 40-dimensional geometric put option studied in Table 4 with  $p = 2$  and test the scalability of our parallel implementation for  $m = 200,000$ . The tests are run on a BullX DLC supercomputer containing 190 nodes for a total of 3204 CPU cores. We report in Table 5 the results of our scalability study using from 1 to 512 cores. Despite the two levels of parallelism available on this supercomputer, we have used a pure MPI implementation without any reference to multithread programming. We could probably have improved the efficiency a bit using two levels of parallelism, but the results are already convincing enough and do not justify the need of a two level approach, which makes the implementation more delicate. The sequential Algorithm runs within one hour and a quarter whereas using 512 cores we manage to get the computational time down to a dozen of seconds, which corresponds to a 0.6 efficiency. Considering the so short wall time required by the run on 512 cores, keeping the efficiency at this level represents a great achievement. Note that with 128 cores, the code runs within a minute with an efficiency of three quarters. These experiments prove the impressive scalability of our algorithm.

#processes	time (sec.)	efficiency
1	4365	1
2	2481	0.99
4	1362	0.90
16	282	0.84
32	272	0.75
64	87	0.78
128	52	0.73
256	34	0.69
512	10.7	0.59

Table 5: Scalability of Algorithm 2 on the 40-dimensional geometric put option described above with  $T = 1$ ,  $r = 0.0488$ ,  $K = 100$ ,  $\sigma^j = 0.3$ ,  $\rho = 0.1$ ,  $\delta^j = 0$ ,  $n = 9$ ,  $p = 2$ .

## 6 Conclusion

We have proposed a purely dual algorithm to compute the price of American or Bermudan options using some stochastic optimization tools. The starting point of our algorithm is the use of Wiener chaos expansion to build a finite dimensional vector space of martingales. Then, we rely on a sample average approximation to effectively optimize the coefficients of the expansion. Our algorithm is very fast: for problems up to dimension 5, a price is obtained within a few seconds, which is a tremendous improvement compared to existing purely dual methods. For higher dimensional problems, we can use a very scalable parallel algorithm to tackle very high dimensional problems (40 underlying assets). We can transparently deal with complex path-dependent payoffs without any extra computational cost. Even though, we restricted to a Brownian setting in this work, our approach could easily be extended to jump diffusion models by introducing Poisson chaos expansion, which is linked to Charlier polynomials (see Geiss and Labart [2016]). We believe that our approach could be improved by cleverly reducing the number of terms in the chaos expansion, the computation of which centralizes most of the effort.

## References

- L. Abbas-Turki, S. Vialle, B. Lapeyre, and P. Mercier. Pricing derivatives on graphics processing units using monte carlo simulation. *Concurrency and Computation: Practice and Experience*, 26(9):1679–1697, 2014.
- L. Andersen and M. Broadie. Primal-dual simulation algorithm for pricing multidimensional american options. *Management Science*, 50(9):1222–1234, 2004.
- V. Bally and G. Pages. A quantization algorithm for solving multidimensional discrete-time optimal stopping problems. *Bernoulli*, 9(6):1003–1049, 2003.
- D. Belomestny. Solving optimal stopping problems via empirical dual optimization. *Ann. Appl. Probab.*, 23(5):1988–2019, 2013.
- D. Belomestny, C. Bender, and J. Schoenmakers. True upper bounds for Bermudan products via non-nested Monte Carlo. *Math. Finance*, 19(1):53–71, 2009.
- D. P. Bertsekas. Stochastic optimization problems with nondifferentiable cost functionals. *J. Optimization Theory Appl.*, 12:218–231, 1973.
- S. Boyd, L. Xiao, and A. Mutapcic. Subgradient methods. *lecture notes of EE392o, Stanford University, Autumn Quarter*, 2004:2004–2005, 2003.
- P. Briand and C. Labart. Simulation of BSDEs by Wiener Chaos Expansion. *Annals of Applied Probability*, 24(3):1129–1171, 2014.
- M. Broadie and P. Glasserman. A stochastic mesh method for pricing high-dimensional american options. *Journal of Computational Finance*, 7:35–72, 2004.
- J. F. Carriere. Valuation of the early-exercise price for options using simulations and non-parametric regression. *Insurance: mathematics and Economics*, 19(1):19–30, 1996.



- M. H. A. Davis and I. Karatzas. A deterministic approach to optimal stopping. In *Probability, statistics and optimisation*, Wiley Ser. Probab. Math. Statist. Probab. Math. Statist., pages 455–466. Wiley, Chichester, 1994.
- V. Dung Doan, A. Gaiwad, M. Bossy, F. Baude, and I. Stokes-Rees. Parallel pricing algorithms for multidimensional bermudan/american options using Monte Carlo methods. *Mathematics and Computers in Simulation*, 81(3):568–577, 2010.
- C. Geiss and C. Labart. Simulation of BSDEs with jumps by wiener chaos expansion. *Stochastic Processes and their Applications*, 2016. URL <http://dx.doi.org/10.1016/j.spa.2016.01.006>.
- F. Jamshidian. The duality of optimal exercise and domineering claims: a Doob-Meyer decomposition approach to the Snell envelope. *Stochastics*, 79(1-2):27–60, 2007.
- B. Jourdain and J. Lelong. Robust Adaptive Importance Sampling for Normal Random Vectors. *Ann. Appl. Probab.*, 19(5):1687–1718, 2009.
- A. Kolodko and J. Schoenmakers. Upper bounds for bermudan style derivatives. *Monte Carlo Methods and Applications mcma*, 10(3-4):331–343, 2004.
- F. Longstaff and R. Schwartz. Valuing American options by simulation : A simple least-square approach. *Review of Financial Studies*, 14:113–147, 2001.
- D. Nualart. Analysis on Wiener space and anticipating stochastic calculus. In B. Springer-Verlag, editor, *Lectures on Probability Theory and Statistics (Saint-Flour, 1995)*, pages 123–227. 1998.
- B. T. Polyak. Introduction to optimization. *Optimization Software*, 1987.
- L. C. G. Rogers. Monte Carlo valuation of American options. *Math. Finance*, 12(3):271–286, 2002.
- L. C. G. Rogers. Dual valuation and hedging of Bermudan options. *SIAM J. Financial Math.*, 1:604–608, 2010.
- R. Y. Rubinstein and A. Shapiro. *Discrete event systems*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Ltd., Chichester, 1993. ISBN 0-471-93419-4. Sensitivity analysis and stochastic optimization by the score function method.
- J. Schoenmakers. *Robust Libor modelling and pricing of derivative products*. CRC Press, 2005.
- J. Schoenmakers, J. Zhang, and J. Huang. Optimal dual martingales, their analysis, and application to new algorithms for bermudan products. *SIAM Journal on Financial Mathematics*, 4(1):86–116, 2013.
- J. A. Tilley. Valuing american options in a path simulation model. *Transactions of the Society of Actuaries*, 45(83):104, 1993.
- J. Tsitsiklis and B. V. Roy. Regression methods for pricing complex American-style options. *IEEE Trans. Neural Netw.*, 12(4):694–703, 2001.